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Completeness of Bethe Ansatz by Sklyanin SOV for Cyclic Representations of Integrable Quantum Models

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Abstract

In [1] an integrable quantum model was introduced and a class of its cyclic representations was proven to define lattice regularizations of the Sine-Gordon model. Here, we analyze general cyclic representations of this integrable quantum model by extending the spectrum construction introduced in [2] in the framework of the Separation of Variables (SOV) of Sklyanin. We show that as in [1] also for general representations, the transfer matrix spectrum (eigenvalues and eigenstates) is completely characterized in terms of polynomial solutions of an associated functional Baxter equation. Moreover, we prove that the method here developed has two fundamental built-in features: i) the completeness of the set of the transfer matrix eigenstates constructed from the solutions of the associated Bethe ansatz equations, ii) the existence and complete characterization of the Baxter Q-operator.

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1. Introduction

In [1] an integrable quantum model was introduced¹ and a class of its cyclic representations was proven to define lattice regularizations of the Sine-Gordon model² in a given sector of the quantum theory. Main results of that paper are the simplicity and the completeness of the characterization of the transfer matrix spectrum.

Here, some comments are probably in order to better point out the relevance of these results.

Integrability: It is of common use to call integrable a quantum model which admits a one parameter family of commuting conserved charges. This is the case in the framework of the quantum inverse scattering method [9, 10, 3] where the transfer matrix defines the family of commuting conserved charges. Anyhow, it is worth remarking that a more precise definition of quantum integrability requires that this family of commuting charges is a complete set of observables; i.e. its spectrum has to be simple (non-degenerate). This last statement has to be independently proven as it was done in [1] for the class of cyclic representations there analyzed.

The first fundamental task to solve for a given integrable quantum model is the exact solution of its spectral problem, i.e. the determination of the eigenvalues and the simultaneous eigenstates of the complete set of its commuting conserved charges.

Completeness of Bethe Ansatz: In the framework of the Bethe ansatz several methods has been introduced to analyze this spectral problem which lead to the characterization of the eigenvalues (and for some of those to the eigenstates) in terms of solutions to an associated system of *Bethe ansatz equations*. Hereafter, a rigorous proof of the completeness of the spectrum characterization was a fundamental and complicate long standing goal³ in the literature. In [1] this result has been reached showing in particular that the eigenstates of the transfer matrix constructed from the solutions of an associated system of Bethe ansatz equations form a basis of the representation. Note that early there were only a few examples of integrable quantum models where the completeness has been proven, including the XXX Heisenberg model; see [14] and references therein.

There is a double motivation for the present article. On the one hand, we are interested in showing that the strong results of simplicity and completeness of the transfer matrix spectrum can be proven for general cyclic representations of the SG model. On the other hand, we want to use these representations to define a method to characterize the spectrum which is based only on the Separation of Variables (SOV) of Sklyanin [15, 16, 17]. Then, these general representations

¹From here on we will refer to it as SG model.

²This lattice regularization goes back to [3, 4, 5] and it is related to formulations which have more recently been studied in [6, 7, 8].

³It was addressed also by numerical methods like the one introduced in [11, 12], see also [13] for an application of it.

are of special interest as they define concrete examples of cyclic representations of quantum integrable models to which the *standard* construction⁴ [24]-[28] of the Baxter Q-operator by using cyclic dilogarithm functions does not apply in general.

1.1 Methodological aim

In the framework of quantum integrability, there are several methods to analyze the spectral problem as the coordinate Bethe ansatz [29, 30, 31], the TQ method [30], the algebraic Bethe ansatz (ABA) [9, 10, 3], the analytic Bethe ansatz [32]. However, they suffer in general from one or more of the following problems: i) *Reduced applicability*; i.e. there exist important examples of quantum integrable models to which these methods do not apply. ii) *Analysis reduced only to the set of eigenvalues*; i.e. they do not allow for the construction of the eigenstates. iii) *Lack of completeness proof*; i.e. the completeness of the spectrum description is not assured by the method while there are known cases for which these methods lead to incomplete spectrum description [33].

The separation of variables (SOV) method of Sklyanin is a more promising approach. Indeed, the SOV method resolves problems like the reduced applicability of other methods; in particular, it works for integrable quantum models to which ABA does not apply. Moreover, it leads to both the eigenvalues and the eigenstates of the transfer matrix with a construction which has as built-in feature the completeness of the spectrum description.

In the case of cyclic representations [34] of integrable quantum models, however, it is worth pointing out that the SOV method leads to the characterization of the transfer matrix spectrum in terms of solutions of an associated finite system⁵ of Baxter-like equations. Thus, SOV method does not lead directly to the standard spectrum characterization expected in the framework of Bethe ansatz approaches. Moreover, such SOV characterization of the spectrum is not the most efficient in view of the analysis of the continuum limit. In order to solve these problems it is important to prove that the SOV characterization can be reformulated in terms of an associated functional Baxter equation. One possibility is to add to the SOV characterization the Q-operator approach. In particular, this can be done when the corresponding Baxter equation is proven to be compatible with the finite system of Baxter-like equations of the SOV characterization. In fact, this was the strategy followed in [1] for the special cyclic representations there analyzed.

The main methodological aim of the present article is to present an approach which allows to reduce the solution of the spectral problem to the classification of the solutions of an associated

⁴Here, we refer to the Baxter's construction by *gauge* transformations which leave unchanged the transfer matrix while make triangular the action of the Lax matrices on the Q-operator. Let us recall that there are also others constructions of the Q-operator; interesting examples are presented in [18]-[23].

⁵The number of equations in the system is finite and related to the dimension of the cyclic representation.

Baxter equation in a fixed class of functions. An approach where the SOV characterization of spectrum is the starting point and the *standard* construction by *gauge* transformations⁶ of the Q-operator is completely bypassed. Let us recall that such an approach has been first developed in [2] for the special class of cyclic representations studied also in [1]. However, in [2] we have used some shortcuts to define the coefficients of the associated functional Baxter equations which hold only for such a special class of cyclic representations. Here, we present this approach in the framework of the general cyclic representations also to infer some universal features of the method for applications to others integrable quantum models.

1.2 Organization of the paper

In Section 2, it is recalled the definition of the SG quantum model in the framework of the quantum inverse scattering method. In Section 3, the construction of the SOV representations corresponding to general cyclic representations of the SG model is implemented. There, the result derived in [1] is extended and completed with the characterization of the coefficients of the SOV representations. Section 4 is the core of the article; there it is explained the method which allows to reformulate the SOV characterization of the spectrum in terms of an associated Baxter functional equation. Moreover, there it is proven the existence of all the polynomial solutions of this equation required to completely characterize the transfer matrix spectrum (eigenvalues and eigenstates) of the SG model. Appendix A and B contains some important properties. In Appendix C, the comparison between our SG model and the $\tau^{(2)}$ -model is made, pointing out the difference in the spectrum of the two models. In particular, it is shown as for even chains the transfer matrices are not similar for general representations.

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⁶Note that this method can be applied only when the existence of some model dependent *quantum dilogarithm* functions [35]-[45] is proven.

2. Lattice SG quantum model

2.1 Definitions and first properties

The Lax operator of the SG model, as introduced in [1], reads:

$$L_n^{\text{SG}}(\lambda) = \frac{\kappa_n}{i} \begin{pmatrix} i u_n (q^{-\frac{1}{2}} \kappa_n v_n + q^{+\frac{1}{2}} \kappa_n^{-1} v_n^{-1}) & \lambda_n v_n - \lambda_n^{-1} v_n^{-1} \\ \lambda_n v_n^{-1} - \lambda_n^{-1} v_n & i u_n^{-1} (q^{+\frac{1}{2}} \kappa_n^{-1} v_n + q^{-\frac{1}{2}} \kappa_n v_n^{-1}) \end{pmatrix}, \quad (2.1)$$

where $\lambda_n \equiv \lambda/\xi_n$ for any $n \in \{1, \dots, N\}$ with ξ_n and κ_n parameters of the representation. For any $n \in \{1, \dots, N\}$ the couple of operators (u_n, v_n) define a Weyl algebra \mathcal{W}_n :

$$u_n v_m = q^{\delta_{nm}} v_m u_n, \quad \text{where } q = e^{-\pi i \beta^2}. \quad (2.2)$$

We will restrict our attention to the case in which q is a p -root of unity:

$$\beta^2 = \frac{p'}{p}, \quad p \equiv 2l + 1, p' \equiv 2l' \quad \text{and} \quad l, l' \in \mathbb{Z}^{>0} \rightarrow q^p = 1. \quad (2.3)$$

In this case each Weyl algebra \mathcal{W}_n admits a finite-dimensional representation of dimension p . Let us denote with:

$$|z\rangle \equiv |z_1, \dots, z_N\rangle \quad \text{with } z_i \in \mathbb{S}_p \equiv \{q^{2n}; n = 0, \dots, 2l\} \quad \text{and } i \in \{1, \dots, N\}, \quad (2.4)$$

the generic state of the basis constructed by the tensor product of the eigenstates $|z_n\rangle$ of each operator u_n , then in this basis the Weyl algebra representation reads:

$$\begin{aligned} v_n |z_1, \dots, z_N\rangle &= v_n |z_1, \dots, q z_n, \dots, z_N\rangle, \\ u_n |z_1, \dots, z_N\rangle &= u_n z_n |z_1, \dots, z_N\rangle. \end{aligned} \quad (2.5)$$

where u_n and v_n are other parameters of the representation.

The monodromy matrix of the model is defined in terms of the Lax operators by:

$$M(\lambda) \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \equiv L_N(\lambda) \dots L_1(\lambda), \quad (2.6)$$

and it satisfies the quadratic relations:

$$R(\lambda/\mu) (M(\lambda) \otimes 1) (1 \otimes M(\mu)) = (1 \otimes M(\mu)) (M(\lambda) \otimes 1) R(\lambda/\mu), \quad (2.7)$$

w.r.t. the six-vertex R -matrix:

$$R(\lambda) = \begin{pmatrix} q\lambda - q^{-1}\lambda^{-1} & & & \\ & \lambda - \lambda^{-1} & q - q^{-1} & \\ & q - q^{-1} & \lambda - \lambda^{-1} & \\ & & & q\lambda - q^{-1}\lambda^{-1} \end{pmatrix}. \quad (2.8)$$

Then the elements of $M(\lambda)$ generate a representation \mathcal{R}_N of the so-called Yang-Baxter algebra characterized by the $4N$ parameters $\kappa = (\kappa_1, \dots, \kappa_N)$, $\xi = (\xi_1, \dots, \xi_N)$, $u = (u_1, \dots, u_N)$ and $v = (v_1, \dots, v_N)$. In particular, the commutation relations (2.7) lead to the mutual commutativity of the elements of the one parameter family of operators:

$$T(\lambda) = \text{tr}_{\mathbb{C}^2} M(\lambda), \quad (2.9)$$

known as transfer matrix.

In [1] the spectral problem of this transfer matrix has been solved for *untwisted* representations:

$$\kappa_n^2 \in \mathbb{R}, \quad \xi_n^2 \in \mathbb{R}, \quad u_n^{2p} = 1, \quad v_n^{2p} = 1, \quad \forall n \in \{1, \dots, N\}. \quad (2.10)$$

In the present paper we will extend the analysis considering general representations:

$$\kappa_n^2 \in \mathbb{R}, \quad \xi_n^2 \in \mathbb{R}, \quad (u_n)^* u_n = 1, \quad (v_n)^* v_n = 1, \quad \forall n \in \{1, \dots, N\}, \quad (2.11)$$

which we call *twisted* for $u_n^{2p} \neq 1$ and $v_n^{2p} \neq 1$.

Let us comment that the above restrictions on the parameters of the representation are imposed to get the self-adjointness of the transfer matrix:

Lemma 1. *If the parameters of the representation satisfy the constraints (2.11) and*

$$\varepsilon \equiv -(\kappa_n \xi_n) / (\kappa_n^* \xi_n^*) \quad \text{is uniform along the chain}, \quad (2.12)$$

then the generators of the Yang-Baxter algebra satisfy the following transformations under Hermitian conjugation:

$$M(\lambda)^\dagger \equiv \begin{pmatrix} A^\dagger(\lambda) & B^\dagger(\lambda) \\ C^\dagger(\lambda) & D^\dagger(\lambda) \end{pmatrix} = \begin{pmatrix} D(\lambda^*) & C(\varepsilon \lambda^*) \\ B(\varepsilon \lambda^*) & A(\lambda^*) \end{pmatrix}, \quad (2.13)$$

which, in particular, imply the self-adjointness of the transfer matrix $T(\lambda)$ for real λ .

Proof. It is simple to observe that the Lax operator of the SG model satisfies the equation (2.13), which can be also written as:

$$(L_n^{\text{SG}}(\lambda))^\dagger = \sigma_1 L_n^{\text{SG}}(\varepsilon \lambda^*) \sigma_1 \xrightarrow{(2.12)} (M(\lambda))^\dagger = \sigma_1 M(\varepsilon \lambda^*) \sigma_1, \quad (2.14)$$

that is (2.13) holds, when one takes into account that $A(\lambda)$ and $D(\lambda)$ are even in λ . \square

In the case of a lattice with N even quantum sites, we can introduce the operator:

$$\Theta = \prod_{n=1}^N v_n^{(-1)^{1+n}}, \quad (2.15)$$

which plays the role of a *grading operator* in the Yang-Baxter algebra:

Lemma 2. (Proposition 6 of [1]) Θ commutes with the transfer matrix and satisfies the following commutation relations with the entries of the monodromy matrix:

$$\Theta C(\lambda) = qC(\lambda)\Theta, \quad [A(\lambda), \Theta] = 0, \quad (2.16)$$

$$B(\lambda)\Theta = q\Theta B(\lambda), \quad [D(\lambda), \Theta] = 0. \quad (2.17)$$

Moreover, the Θ -charge allows to express the asymptotics of the transfer matrix as:

$$\lim_{\log \lambda \rightarrow \mp \infty} \lambda^{\pm N} T(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a \xi_a^{\pm 1}}{i} \right) (\Theta + \Theta^{-1}). \quad (2.18)$$

Let us denote with Σ_T the set of the eigenvalue functions $t(\lambda)$ of the transfer matrix $T(\lambda)$. By the definitions (2.1) and (2.9), Σ_T is a subset of $\mathbb{R}[\lambda^2, \lambda^{-2}]_{\bar{N}/2}$, where we are using the notations:

$$\bar{N} \equiv N + e_N - 1, \quad \text{and } e_N = 0 \text{ for } N \text{ odd and } 1 \text{ for } N \text{ even}, \quad (2.19)$$

and $\mathbb{R}[x, x^{-1}]_M$ denotes the linear space in the field \mathbb{R} of the *real*:

$$f(x) \in \mathbb{R}[x, x^{-1}]_M \rightarrow (f(x))^* = f(x^*) \quad \forall x \in \mathbb{C}, \quad (2.20)$$

Laurent polynomials of degree M in the variable x .

Note that in the case of N even, the Θ -charge naturally induces the grading $\Sigma_T = \bigcup_{k=0}^{2l} \Sigma_T^{\theta, k}$, where:

$$\Sigma_T^{\theta, k} \equiv \left\{ t(\lambda) \in \Sigma_T : \lim_{\log \lambda \rightarrow \mp \infty} \lambda^{\pm N} t(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a \xi_a^{\pm 1}}{i} \right) (q^k \theta + (q^k \theta)^{-1}) \right\}. \quad (2.21)$$

This simply follows by the asymptotics of $T(\lambda)$ and by its commutativity with Θ . In particular, any $t(\lambda) \in \Sigma_T^{\theta, k}$ is a T -eigenvalue corresponding to simultaneous eigenstates of $T(\lambda)$ and Θ with Θ -eigenvalue $q^k \theta$.

3. Cyclic SOV representations

The separation of variables (SOV) of Sklyanin [15]-[17] is a method to solve the spectral problem for $T(\lambda)$ which is based on the observation that such a problem simplifies considerably if one works in a SOV representation: A representation where the commutative family of operators $B(\lambda)$ is diagonal. This is due to the simple form assumed by the operator families $A(\lambda)$ and $D(\lambda)$ in such a representation.

3.1 Cyclic SOV representations: Generality

In subsection 3.3 we will show that the commuting family of operators⁷ $B_N(\lambda)$ is diagonalizable with simple spectrum for almost all the values of the $4N$ parameters κ, ξ, u and v of our model. The corresponding SOV representations of the SG model are to a large extent determined by the Yang-Baxter algebra (2.7); here we recall the general form of these representations.

Let $\langle \eta |$ be the generic element of a basis of eigenvectors of $B_N(\lambda)$:

$$\langle \eta | B_N(\lambda) = \eta_N^{e_N} b_\eta(\lambda) \langle \eta |, \quad b_\eta(\lambda) \equiv \prod_{n=1}^N \frac{\kappa_n}{i} \prod_{a=1}^{[N]} (\lambda/\eta_a - \eta_a/\lambda), \quad (3.1)$$

where $[N] \equiv N - e_N$ and

$$\eta \in \mathbb{B}_N \equiv \{ (q^{k_1} \zeta_1, \dots, q^{k_N} \zeta_N); (k_1, \dots, k_N) \in \mathbb{Z}_p^N \}. \quad (3.2)$$

Here, the simplicity of the spectrum of $B_N(\lambda)$ is equivalent to the requirement $\zeta_a^p \neq \zeta_b^p$ for any $a \neq b \in \{1, \dots, [N]\}$. The remaining generators of the Yang-Baxter algebra read:

$$A_N(\lambda) = e_N b_\eta(\lambda) \left[\frac{\lambda}{\eta_A} T_N^+ - \frac{\eta_A}{\lambda} T_N^- \right] + \sum_{a=1}^{[N]} \prod_{b \neq a} \frac{\lambda/\eta_b - \eta_b/\lambda}{\eta_a/\eta_b - \eta_b/\eta_a} a_N(\eta_a) T_a^-, \quad (3.3)$$

$$D_N(\lambda) = e_N b_\eta(\lambda) \left[\frac{\lambda}{\eta_D} T_N^- - \frac{\eta_D}{\lambda} T_N^+ \right] + \sum_{a=1}^{[N]} \prod_{b \neq a} \frac{\lambda/\eta_b - \eta_b/\lambda}{\eta_a/\eta_b - \eta_b/\eta_a} d_N(\eta_a) T_a^+, \quad (3.4)$$

where T_a^\pm are the operators defined by

$$\langle \eta_1, \dots, \eta_N | T_a^\pm = \langle \eta_1, \dots, q^{\pm 1} \eta_a, \dots, \eta_N |.$$

While, $C_N(\lambda)$ is uniquely⁸ defined by the quantum determinant relation:

$$\det_q M(\lambda) \equiv A(\lambda) D(q^{-1} \lambda) - B(\lambda) C(q^{-1} \lambda), \quad (3.5)$$

where $\det_q M(\lambda)$ is a central element⁹ of the Yang-Baxter algebra (2.7) which reads:

$$\det_q M(\lambda) \equiv \prod_{n=1}^N \kappa_n^2 (\lambda/\mu_{n,+} - \mu_{n,+}/\lambda) (\lambda/\mu_{n,-} - \mu_{n,-}/\lambda), \quad (3.6)$$

⁷From here, we will use the index N when it will be need to point out that we are referring to the chain with N sites and we will omit it otherwise.

⁸Note that the operator $B_N(\lambda)$ is invertible except for λ which coincides with a zero of B_N , so in general $C_N(\lambda)$ is defined by (4.5) just inverting $B_N(\lambda)$. This is enough to fix in an unique way the operator C_N being it a Laurent polynomial of degree $[N]$ in λ .

⁹The centrality of the quantum determinant in the Yang-Baxter algebra was first discovered in [46]; see also [47] for an historical note.

where $\mu_{n,\pm} \equiv \pm i\kappa_n^{\pm 1} q^{1/2} \xi_n$. Remark that the quantum determinant does not depend from the parameters u_n and v_n of the representation and so it is the same for twisted and untwisted representations.

The expressions (3.3) and (3.4) contain complex-valued coefficients $\eta_A, \eta_D, a_N(\eta_r)$ and $d_N(\eta_r)$. Note that $Z_A \equiv \eta_A^p$ and $Z_D \equiv \eta_D^p$ are fixed by the asymptotics relations (B.9) while the coefficients $a_N(\eta_r)$ and $d_N(\eta_r)$ are restricted by the quantum determinant condition:

$$\det_q M(\eta_r) = a_N(\eta_r) d_N(q^{-1} \eta_r), \quad \forall r = 1, \dots, [N]. \quad (3.7)$$

In a SOV representation, some freedom is left in the choice of $a_N(\eta_r)$ and $d_N(\eta_r)$ which can be parametrized by what we call a gauge transformation:

$$a'_N(\eta_r) = a_N(\eta_r) \frac{f(\eta_r q^{-1})}{f(\eta_r)}, \quad d'_N(\eta_r) = d_N(\eta_r) \frac{f(\eta_r q)}{f(\eta_r)}; \quad (3.8)$$

which just amounts in a renormalization in the states of the B-eigenbasis:

$$\langle \eta | \rightarrow \prod_{r=1}^N f^{-1}(\eta_r) \langle \eta |. \quad (3.9)$$

3.2 Central elements: Average values

Following [34], let us define the average value \mathcal{O} of the elements of the monodromy matrix $M(\lambda)$ as

$$\mathcal{O}(\Lambda) = \prod_{k=1}^p \mathcal{O}(q^k \lambda), \quad \Lambda = \lambda^p, \quad (3.10)$$

where \mathcal{O} can be A_N, B_N, C_N or D_N and we have to remark that the commutativity of each family of operators $A_N(\lambda), B_N(\lambda), C_N(\lambda)$ and $D_N(\lambda)$ implies that the corresponding average values are functions of Λ . So that $\mathcal{A}_N(\Lambda), \mathcal{D}_N(\Lambda)$ are even Laurent polynomials of degree \bar{N} while $\mathcal{B}_N(\Lambda), \mathcal{C}_N(\Lambda)$ are odd Laurent polynomials of degree $[N]$ in Λ .

Proposition 1.

a) *The average values $\mathcal{A}_N(\Lambda), \mathcal{B}_N(\Lambda), \mathcal{C}_N(\Lambda), \mathcal{D}_N(\Lambda)$ of the monodromy matrix elements are central elements which satisfy the following relations:*

$$(\mathcal{A}_N(\Lambda))^* \equiv \mathcal{D}_N(\Lambda^*), \quad (\mathcal{B}_N(\Lambda))^* \equiv \mathcal{C}_N(\varepsilon \Lambda^*), \quad (3.11)$$

under complex conjugation.

b) Let $\mathcal{M}(\Lambda)$ be the 2×2 matrix with elements the average values of the elements of the monodromy matrix $M(\lambda)$, then it holds:

$$\mathcal{M}_N(\Lambda) = \mathcal{L}_N(\Lambda) \mathcal{L}_{N-1}(\Lambda) \dots \mathcal{L}_1(\Lambda). \quad (3.12)$$

where $\mathcal{L}_n(\Lambda)$ is the 2×2 matrix with elements the average values of the elements of the Lax matrix $L_n^{SG}(\lambda)$.

A similar statement was first proven in [34].

Proof of a). Centrality of $\mathcal{B}_N(\Lambda)$ trivially follows from the fact that $B_N(\lambda)$ is diagonal in the SOV representation and from (3.1) it is easily found:

$$\mathcal{B}_N(\Lambda) = Z_N^{\text{en}} \prod_{n=1}^N \frac{K_n}{i^p} \prod_{a=1}^{[N]} (\Lambda/Z_a - Z_a/\Lambda), \quad \begin{aligned} Z_a &\equiv \eta_a^p, \\ K_a &\equiv \kappa_a^p. \end{aligned} \quad (3.13)$$

The requirement of cyclicity of the SOV representation reads:

$$(\mathcal{T}_a^-)^p = (\mathcal{T}_a^+)^p = 1 \quad \forall a \in \{1, \dots, N\}, \quad (3.14)$$

so that $\mathcal{A}_N(Z_r)$ and $\mathcal{D}_N(Z_r)$ are centrals and related to the coefficients $a_N(q^k \eta_r)$ and $d_N(q^k \eta_r)$ by

$$\mathcal{A}_N(Z_r) \equiv \prod_{k=1}^p a_N(q^k \eta_r), \quad \mathcal{D}_N(Z_r) \equiv \prod_{k=1}^p d_N(q^k \eta_r), \quad \forall r \in \{1, \dots, [N]\}. \quad (3.15)$$

Note that $\mathcal{A}_N(\Lambda) \Lambda^{\bar{N}}$ and $\mathcal{D}_N(\Lambda) \Lambda^{\bar{N}}$ are polynomials in Λ^2 of degree \bar{N} . So the centrality of $\mathcal{A}_N(\Lambda)$ and $\mathcal{D}_N(\Lambda)$ follows from the simplicity of the B_N -spectrum and the centrality in the special values (3.15) to which we have to add for even N the centrality of the leading asymptotic terms of $\mathcal{A}_N(\lambda)$ and $\mathcal{D}_N(\lambda)$ as discussed in appendix B. Finally, the Hermitian conjugation properties of the elements of the monodromy matrix:

$$(\mathcal{A}_N(\lambda))^{\dagger} \equiv \mathcal{D}_N(\lambda^*), \quad (\mathcal{B}_N(\lambda))^{\dagger} \equiv \mathcal{C}_N(\varepsilon \lambda^*), \quad (3.16)$$

imply (3.11) and in particular the centrality of $\mathcal{C}_N(\Lambda)$. \square

Proof of b). Under the assumption that $B(\lambda)$ is diagonalizable and with simple spectrum in the entire chain as well as in each subchain, the point b) follows inductively by using the next Lemma. \square

Lemma 3. (Proposition 3 of [1]) *The following recursive equations on the average values hold:*

$$\mathcal{B}_N(\Lambda) = \mathcal{A}_M(\Lambda) \mathcal{B}_{N-M}(\Lambda) + \mathcal{B}_M(\Lambda) \mathcal{D}_{N-M}(\Lambda), \quad (3.17)$$

$$\mathcal{C}_N(\Lambda) = \mathcal{D}_M(\Lambda) \mathcal{C}_{N-M}(\Lambda) + \mathcal{C}_M(\Lambda) \mathcal{A}_{N-M}(\Lambda), \quad (3.18)$$

$$\mathcal{A}_N(\Lambda) = \mathcal{A}_M(\Lambda) \mathcal{A}_{N-M}(\Lambda) + \mathcal{B}_M(\Lambda) \mathcal{C}_{N-M}(\Lambda), \quad (3.19)$$

$$\mathcal{D}_N(\Lambda) = \mathcal{D}_M(\Lambda) \mathcal{D}_{N-M}(\Lambda) + \mathcal{C}_M(\Lambda) \mathcal{B}_{N-M}(\Lambda), \quad (3.20)$$

where on the l.h.s. there are average values of the monodromy matrix elements on the complete chain with N -sites while on the r.h.s. there are those of the monodromy matrices on the subchains 1 and 2 with $(N - M)$ -sites and M -sites, respectively.

Remark 1. Being the average values central elements of the representation they are unchanged by similarity transformations and they therefore represent parameters of the representation. Moreover, the gauge transformations clearly leave Z_r , Z_A , Z_D , $\mathcal{A}_N(Z_r)$ and $\mathcal{D}_N(Z_r)$ unchanged. Therefore these last numbers characterize gauge-invariant dates of the SOV representations.

3.2.1 Calculation of average values

It is a simple exercise to show that the average values of the elements of the Lax matrices $L_n^{\text{SG}}(\lambda)$ are explicitly given by

$$\mathcal{L}_n(\Lambda) = \frac{1}{i^p} \begin{pmatrix} i^p U_n (K_n^2 V_n + V_n^{-1}) & K_n (\Lambda V_n / X_n - X_n / V_n \Lambda) \\ K_n (\Lambda / X_n V_n - X_n V_n / \Lambda) & i^p U_n^{-1} (K_n^2 V_n^{-1} + V_n) \end{pmatrix}, \quad (3.21)$$

where we have used the notations $K_n = \kappa_n^p$, $X_n = \xi_n^p$, $U_n = u_n^p$ and $V_n = v_n^p$. This formula together with equality (3.12) allows to uniquely define the average values of the monodromy matrix elements and to state:

Lemma 4. $\mathcal{O}_N(\Lambda) / \prod_{n=1}^N K_n$ for $\mathcal{O} = \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are Laurent polynomials of maximal degree 1 in each of the parameters K_n, X_n, U_n, V_n .

Note that the previous lemma also implies that the gauge-invariant dates of the SOV representations, Z_r , Z_A , Z_D , $\mathcal{A}_N(Z_r)$ and $\mathcal{D}_N(Z_r)$ (up to permutations of $r = 1, \dots, [N]$), are uniquely defined in terms of the parameters of the representation K_n, X_n, U_n, V_n .

3.2.2 Choice of the gauge in the SOV representation

Let us recall that the SOV coefficients $\mathbf{a}(\eta_r)$ and $\mathbf{d}(\eta_r)$ are specified only by the average value relations (3.15) required by the cyclicity of the SOV representation. Then we can fix the gauge in the SOV representation fixing a couple $\mathbf{a}(\lambda)$ and $\mathbf{d}(\lambda)$ of Laurent polynomial solutions of the following average relations:

$$\mathcal{A}(\Lambda) + \gamma \mathcal{B}(\Lambda) = \prod_{k=1}^p \mathbf{a}(q^k \lambda), \quad \mathcal{D}(\Lambda) + \delta \mathcal{B}(\Lambda) = \prod_{k=1}^p \mathbf{d}(q^k \lambda), \quad (3.22)$$

where γ and δ are constants to be fixed. As it will be clear in the next sections, it is important to require that these solutions satisfy the further conditions:

$$(\mathbf{a}(\lambda))^* = \mathbf{d}(\lambda^*), \quad (3.23)$$

$$\text{if } a_0 \in Z_{\mathbf{a}(\lambda)} \rightarrow q^k a_0 \notin Z_{\mathbf{a}(\lambda)} \quad \forall k \in \{1, \dots, 2l\}, \quad (3.24)$$

$$Z_{\mathbf{a}(\lambda)} \cap Z_{\prod_{h=0}^{2l-1} d(\lambda q^h)} = \emptyset, \quad (3.25)$$

where $Z_{f(\lambda)}$ denotes the set of the zeros of the function $f(\lambda)$. Let us denote with:

$$S_{p, \lambda_0} \equiv (\lambda_0, q\lambda_0, \dots, q^{2l}\lambda_0) \in \mathbb{C}^p, \quad \text{a } p\text{-string of center } \lambda_0, \quad (3.26)$$

then condition (3.24), in particular, implies that $\mathbf{a}(\lambda)$ and $\mathbf{d}(\lambda)$ are free from p -strings¹⁰. Note that in the untwisted representations ($U_n = V_n = 1$), we have defined these Laurent polynomials as solutions of (3.22) with $\gamma = -\delta = 1$ which also satisfy (3.23)-(3.25). This was possible as for untwisted representations it holds:

$$\mathcal{A}(\Lambda) = \mathcal{D}(\Lambda), \quad \mathcal{B}(\Lambda) = \mathcal{C}(\Lambda). \quad (3.27)$$

It is worth to note that this properties together with (3.11) imply that for untwisted representations the average values of the elements of the monodromy matrix are *real* Laurent polynomial in Λ . However, this reality condition is lost for the twisted representations here considered; indeed, we have the following:

Lemma 5. *Almost for all the values of the parameters K_n, X_n, U_n, V_n the Laurent polynomials $\mathcal{A}(\Lambda)$ and $\mathcal{D}(\Lambda)$ are not identical as well as $\mathcal{B}(\Lambda)$ and $\mathcal{C}(\Lambda)$.*

Proof. The Lemma is true for $N = 1$, as it is clear from the one-site average formula (3.21). Then, for $N > 1$, the proof follows by induction using Lemma 3. \square

The previous Lemma and the Hermitian conjugation properties (3.11) imply that for generic twisted representations the only way to simultaneously satisfy (3.22) and (3.23) is by imposing $\gamma = \delta = 0$. Then representing:

$$\mathcal{A}_N(\Lambda) \equiv \mathcal{A}_N \prod_{n=1}^{\bar{N}} (\Lambda / Z_n^{\mathcal{A}} - Z_n^{\mathcal{A}} / \Lambda), \quad (3.28)$$

we can chose to define $\mathbf{d}(\lambda)$ by (3.23) and $\mathbf{a}(\lambda)$ as any even Laurent polynomial of the form:

$$\mathbf{a}(\lambda) \equiv \mathbf{a}_N \prod_{n=1}^{\bar{N}} (\lambda / z_n^{\mathbf{a}} - z_n^{\mathbf{a}} / \lambda) \quad \text{with } \mathbf{a}_N^p \equiv \mathcal{A}_N \text{ and } (z_n^{\mathbf{a}})^p \equiv Z_n^{\mathcal{A}}, \quad (3.29)$$

with asymptotics for N even:

$$\lim_{\log \lambda \rightarrow \mp \infty} \lambda^{\pm N} \mathbf{a}(\lambda) = \left(\prod_{a=1}^N i \kappa_a \xi_a^{\pm 1} \right) \theta^{\pm 1} q^{(1 \mp 1)N}, \quad (3.30)$$

where θ is a fixed p -root of the average value of the Θ -charge. Moreover, we impose the following prescriptions:

¹⁰Here and in the following we say that a function $f(\lambda)$ is free from p -strings meaning that the set of its zeros does not contain p -strings, i.e. $S_{p, \lambda_0} \not\subset Z_{f(\lambda)} \quad \forall \lambda_0 \in \mathbb{C}$.

a) In the case of real or imaginary zeros of $\mathcal{A}(\Lambda)$:

$$\text{If } \exists n \in \{1, \dots, \bar{N}\} : Z_n^{\mathcal{A}} \in \mathbf{Z}_{\mathcal{A}(\Lambda)}, (Z_n^{\mathcal{A}})^2 \in \mathbb{R} \rightarrow z_n^{\mathbf{a}} \in \mathbf{Z}_{\mathbf{a}(\lambda)}, (z_n^{\mathbf{a}})^2/q \in \mathbb{R}. \quad (3.31)$$

b) In the case of multiple zeros of $\mathcal{A}(\Lambda)$ with multiplicity r :

$$\text{If } \exists n_1, \dots, n_r \in \{1, \dots, \bar{N}\} : Z_{n_1}^{\mathcal{A}} = \dots = Z_{n_r}^{\mathcal{A}} \in \mathbf{Z}_{\mathcal{A}(\Lambda)} \rightarrow z_{n_1}^{\mathbf{a}} = \dots = z_{n_r}^{\mathbf{a}} \in \mathbf{Z}_{\mathbf{a}(\lambda)}. \quad (3.32)$$

c) In the case of couples of complex conjugate zeros of $\mathcal{A}(\Lambda)$:

$$\text{If } \exists n \neq m \in \{1, \dots, \bar{N}\} : (Z_n^{\mathcal{A}})^* = Z_m^{\mathcal{A}} \in \mathbf{Z}_{\mathcal{A}(\Lambda)} \rightarrow (z_n^{\mathbf{a}}/q)^* = z_m^{\mathbf{a}} \in \mathbf{Z}_{\mathbf{a}(\lambda)}. \quad (3.33)$$

Then under these conditions $\mathbf{a}(\lambda)$ and $\mathbf{d}(\lambda)$ satisfy the requirements (3.23)-(3.25).

3.3 Constructive proof of the existence of cyclic SOV representations

In the following subsections we will show by recursive construction the

Theorem 1. *Almost for all the values of the parameters K_n, X_n, U_n, V_n there exists a SOV representation for the SG model, in fact, we can prove that the one-parameter operator family $\mathbf{B}(\lambda)$ is diagonalizable with simple spectrum.*

3.3.1 Recursive construction of B-eigenstates

We will construct the eigenstates $\langle \eta |$ of $\mathbf{B}(\lambda) \equiv \mathbf{B}_N(\lambda)$ recursively by induction on N .

In the case $N = 1$ we may simply take $\langle \eta_1 | = \langle v_1 |$, where $\langle v_1 |$ is an eigenstate of the operator v_1 with eigenvalue v . It is useful to note that the inhomogeneity parameter determines the subset of \mathbb{C} on which the variable η_1 lives, $\eta_1 \in (\xi_1/v_1)\mathbb{S}_p$.

Let us assume we have constructed the \mathbf{B}_M -eigenbasis for any $M < N$, then the eigenstates $\langle \eta |$ of $\mathbf{B}_N(\lambda)$ may be constructed in the following form

$$\langle \eta | = \sum_{\chi_1} \sum_{\chi_2} K_N(\eta | \chi_2; \chi_1) \langle \chi_2 | \otimes \langle \chi_1 |, \quad (3.34)$$

where $\langle \chi_2 |$ and $\langle \chi_1 |$ are eigenstates of $\mathbf{B}_M(\lambda)$ and $\mathbf{B}_{N-M}(\lambda)$ with eigenvalues parameterized as in (3.1) by the tuples $\chi_2 = (\chi_{2a})_{a=1, \dots, M}$ and $\chi_1 = (\chi_{1a})_{a=1, \dots, N-M}$, respectively. It suffices to consider the cases where $N - M$ is odd.

3.3.1.a Dependence of the kernel $K_N(\eta | \chi_2; \chi_1)$ w.r.t. χ_2 and χ_1 .

From the formula

$$B_N(\lambda) = A_{2-M}(\lambda)B_{1-N-M}(\lambda) + B_{2-M}(\lambda)D_{1-N-M}(\lambda) \quad (3.35)$$

it follows that the matrix elements of the kernel $K_N(\eta | \chi_2; \chi_1)$ have to satisfy the relations

$$\begin{aligned} & (A_{2-M}(\lambda)B_{1-N-M}(\lambda) + B_{2-M}(\lambda)D_{1-N-M}(\lambda))^t K_N(\eta | \chi_2; \chi_1) \\ &= \eta_N^{eN} \prod_{n=1}^N \frac{\kappa_n}{i} \prod_{a=1}^{[N]} (\lambda/\eta_a - \eta_a/\lambda) K_N(\eta | \chi_2; \chi_1), \end{aligned} \quad (3.36)$$

where we used the notation O^t for the transpose of an operator O . Let us assume that

$$\chi_{1a} q^{h_1} \notin Z_{\det_q M_{1,N-M}(\lambda)}, \quad \chi_{2b} q^{h_2} \notin Z_{\det_q M_{2,M}(\lambda)} \quad \text{and} \quad \chi_{1a} q^{h_1} \neq \chi_{2b} q^{h_2}, \quad (3.37)$$

where $h_i \in \{1, \dots, p\}$, $a \in \{1, \dots, N-M\}$ and $b \in \{1, \dots, M\}$. Under these assumptions the previous equations yield recursion relations for the dependence of the kernel in the variables χ_{1a} and χ_{2b} simply by setting $\lambda = \chi_{1a}$ and $\lambda = \chi_{2b}$. Indeed for $\lambda = \chi_{1a}$ the first term on the left of (3.36) vanishes leading to

$$\begin{aligned} & T_{1a}^- K_N(\eta | \chi_2; \chi_1) d_1(q^{-1}\chi_{1a}) \chi_M^{eM} \prod_{n=1}^{N-M} \frac{i}{\kappa_n} \prod_{a=1}^{[M]} (\chi_{1a}/\chi_{2b} - \chi_{2b}/\chi_{1a}) \\ &= K_N(\eta | \chi_2; \chi_1) \eta_N^{eN} \prod_{b=1}^{[N]} (\chi_{1a}/\eta_b - \eta_b/\chi_{1a}), \end{aligned} \quad (3.38)$$

while for $\lambda = \chi_{2a}$ one finds similarly

$$\begin{aligned} & T_{2a}^+ K_N(\eta | \chi_2; \chi_1) a_2(q^{+1}\chi_{2a}) \prod_{n=1}^M \frac{i}{\kappa_n} \prod_{b=1}^{N-M} (\chi_{2a}/\chi_{1b} - \chi_{1b}/\chi_{2a}) \\ &= K_N(\eta | \chi_2; \eta_1) \eta_N^{eN} \prod_{b=1}^{[N]} (\chi_{2a}/\eta_b - \eta_b/\chi_{2a}), \end{aligned} \quad (3.39)$$

where $d_1(\chi_{1a})$ and $a_2(\chi_{2a})$ are the known coefficients of the SOV representations in the subchains 1 and 2. If M is even we find the recursion relation determining the dependence on χ_{2M} by sending $\lambda \rightarrow \infty$ in (3.36), leading to

$$T_{2M}^+ K_N(\eta | \chi_2; \chi_1) \frac{1}{\chi_{2A}} \prod_{a=1}^{M-1} \frac{1}{\chi_{2a}} \prod_{b=1}^{N-M} \frac{1}{\chi_{1b}} = K_N(\eta | \chi_2; \eta_1) \prod_{b=1}^N \frac{1}{\eta_b}. \quad (3.40)$$

3.3.1.b Dependence of the kernel $K_N(\eta | \chi_2; \chi_1)$ w.r.t. η .

The action

$$\langle \eta | D_N(\eta_i) = -\det_q M_{2M}(q\eta_i) \langle \eta | B_{2M}^{-1}(q\eta_i) B_{1N-M}(\eta_i), \quad (3.41)$$

follows from the formula

$$D_N(\lambda) = D_{2M}(\lambda) D_{1N-M}(\lambda) + C_{2M}(\lambda) B_{1N-M}(\lambda) \quad (3.42)$$

when we express $C_{2M}(\lambda)$ by the quantum determinant $\det_q M_{2M}(\lambda)$ in the subchain 2 and we use that

$$\langle \eta | B_{1N-M}(\eta_i) = -\langle \eta | D_{1N-M}(\eta_i) A_{2M}^{-1}(q\eta_i) B_{2M}(q\eta_i). \quad (3.43)$$

So the kernel matrix elements $K_N(\eta | \chi_2; \chi_1)$ have to satisfy the following recursion relations for the dependence w.r.t. η

$$d_N(\eta_i) T_i^+ K_N(\eta | \chi_2; \chi_1) = -\frac{K_N(\eta | \chi_2; \chi_1) \det_q M_{2M}(q\eta_i) \prod_{a=1}^{N-M} (\kappa_a/i) (\eta_i/\chi_{1a} - \chi_{1a}/\eta_i)}{\chi_{2M}^{e_M} \prod_{a=N-M+1}^N \kappa_a/i \prod_{a=1}^{[M]} (q\eta_i/\chi_{2a} - \chi_{2a}/q\eta_i)}. \quad (3.44)$$

If N is even we have to consider further the asymptotic of the equation (3.42) which by the asymptotic behavior, analyzed in appendix B, leads to the identity:

$$\lim_{\log \lambda \rightarrow \mp \infty} \lambda^{\pm N} D_N(\lambda) = -\lim_{\log \lambda \rightarrow \mp \infty} \lambda^{\pm N} \det_q M_{2M}(\lambda q) B_{2M}^{-1}(\lambda q) B_{1N-M}(\lambda). \quad (3.45)$$

From which we get the following recursion relation determining the kernel dependence on η_N :

$$T_N^+ K_N(\eta | \chi_2; \chi_1) \eta_D \prod_{a=1}^{N-1} \eta_a = K_N(\eta | \chi_2; \eta_1) \prod_{b=1}^M \frac{\xi_{N-M+b}^2}{\chi_{2b}} \prod_{b=1}^{N-M} \chi_{1b}. \quad (3.46)$$

3.3.2 Determination of gauge-invariant SOV dates: $Z_r, Z_A, Z_D, \mathcal{A}_N(Z_r), \mathcal{D}_N(Z_r)$

The condition of compatibility among the recursion relations (3.38)-(3.39) and (3.40) and the requirement of cyclicity, $(T_{1a}^-)^p = (T_{2a}^+)^p = 1$, provide a system of N algebraic equations in the N unknown $Z_a \equiv \eta_a^p$:

$$\mathcal{D}_{1N-M}(\chi_{1a}^p) (\chi_{2M}^{e_M})^p \prod_{n=1}^{N-M} \frac{i^p}{\kappa_n^p} \prod_{b=1}^{[M]} (\chi_{1a}^p/\chi_{2b}^p - \chi_{2b}^p/\chi_{1a}^p) = (\eta_N^{e_N})^p \prod_{b=1}^{[N]} (\chi_{1a}^p/\eta_b^p - \eta_b^p/\chi_{1a}^p), \quad (3.47)$$

$$\mathcal{A}_{2M}(\chi_{2a}^p) \prod_{n=1}^M \frac{i^p}{\kappa_n^p} \prod_{b=1}^{N-M} (\chi_{2a}^p/\chi_{1b}^p - \chi_{1b}^p/\chi_{2a}^p) = (\eta_N^{e_N})^p \prod_{b=1}^{[N]} (\chi_{2a}^p/\eta_b^p - \eta_b^p/\chi_{2a}^p), \quad (3.48)$$

with for M even

$$\frac{1}{\chi_{2A}^p} \prod_{a=1}^{M-1} \frac{1}{\chi_{2a}^p} \prod_{b=1}^{N-M} \frac{1}{\chi_{1b}^p} = \prod_{b=1}^N \frac{1}{\eta_b^p}. \quad (3.49)$$

The simplicity of the spectrum of $B(\lambda)$ in the SOV representations of both the subchains 1 and 2 implies that the above system of equations completely determines the unknown Z_a in terms of χ_{2a}^p, χ_{1a}^p . Indeed, we can reformulate this system of equations in terms of the following Laurent polynomial equation:

$$\mathcal{A}_M(\Lambda)\mathcal{B}_{N-M}(\Lambda) + \mathcal{B}_M(\Lambda)\mathcal{D}_{N-M}(\Lambda) = Z_N^{\text{en}} \prod_{n=1}^N \frac{K_n}{i^p} \prod_{a=1}^{[N]} (\Lambda/Z_a - Z_a/\Lambda), \quad (3.50)$$

where the l.h.s. is formed out of the known average values of the monodromy matrix elements in the subchains 1 and 2. Therefore the problem to determine the unknown Z_a for $a \in \{1, \dots, [N]\}$ is reduced to the problem to determine the zeros of the known Laurent polynomial at the l.h.s. of (3.50).

The requirement of cyclicity $(T_i^+)^p = 1$ determines the remaining gauge-invariant SOV dates:

$$\mathcal{D}_N(Z_i) = -\det_q \mathcal{M}_M(Z_i) \frac{\mathcal{B}_{N-M}(Z_i)}{\mathcal{B}_M(Z_i)}, \quad \mathcal{A}_N(Z_i) = -\det_q \mathcal{M}_{N-M}(Z_i) \frac{\mathcal{B}_M(Z_i)}{\mathcal{B}_{N-M}(Z_i)}, \quad (3.51)$$

for any $i \in \{1, \dots, [N]\}$ where we have defined:

$$\det_q \mathcal{M}_X(\Lambda) \equiv \prod_{a=1}^p \det_q M_X(q^a \lambda), \quad X = M, N-M, N, \quad (3.52)$$

while for N even the condition $(T_N^+)^p = 1$ reads:

$$Z_D = \prod_{a=1}^{N-1} \frac{1}{Z_a} \prod_{b=1}^{N-M} \chi_{1b}^p \prod_{b=1}^N \frac{X_{N-M+b}^2}{\chi_{2b}^p}. \quad (3.53)$$

Note that from the identity:

$$\det_q \mathcal{M}_X(\Lambda) = \mathcal{A}_X(\Lambda)\mathcal{D}_X(\Lambda) - \mathcal{B}_X(\Lambda)\mathcal{C}_X(\Lambda), \quad X = M, N-M, \quad (3.54)$$

the equations (3.51) and (3.53) are equivalent to the equations (3.19) and (3.20) as well as the equation (3.50) coincides with (3.17) of Lemma 3.

3.3.3 Completeness of B-eigenstates and simplicity of B-spectrum

Here we show that the set of B-eigenstates $\langle \eta |$ constructed in the previous subsection is complete, i.e. it defines a basis of the representation; this is done showing that there are p^N distinct corresponding B-eigenvalues. In particular, we can prove:

Proposition 2. *The SOV dates Z_r with $r \in \{1, \dots, [N]\}$ are all distinct for almost all the values of the parameters K_n, X_n, U_n, V_n of the SG model.*

Proof. Note that we can apply Lemma 4 to the average values on the l.h.s. of (3.50) and so we have that the r.h.s. of (3.50) times $\prod_{n=1}^N K_n^{-1}$ is a Laurent polynomial of maximal degree 1 in each of the parameters K_n, X_n, U_n, V_n . The consequent functional dependence¹¹ of the $Z_1, \dots, Z_{[N]}$ w.r.t. these parameters implies that it is sufficient to show the non-vanishing of the Jacobian:

$$J(X; K, U, V) \equiv \det \left(\frac{\partial Z_r}{\partial X_s} \right)_{r,s=1,\dots,[N]} \neq 0 \quad (3.55)$$

for some special values of the parameters K_n, U_n, V_n in order to prove that $J(X; K, U, V) \neq 0$ for almost all the values. Whenever $J(X; K, U, V) \neq 0$, we have invertibility of the map $Z = Z(X_1, \dots, X_{[N]})$ from which the claim of the Proposition follows.

To show that (3.55) is indeed satisfied, let us choose $K_n/V_n = \pm i^p$ for $n = 1, \dots, [N]$, then the average values (3.21) of the Lax operators simplify to

$$\mathcal{L}_n(\Lambda) = \pm \begin{pmatrix} 0 & V_n(\Lambda V_n/X_n - X_n/V_n \Lambda) \\ V_n(\Lambda/V_n X_n - X_n V_n/\Lambda) & 0 \end{pmatrix}. \quad (3.56)$$

Using now (3.12) to compute the l.h.s. of (3.50) we get:

$$[U_N(K_N^2 V_N + V_N^{-1})]^{en} \prod_{n=1}^{[N]} (\Lambda V_n^{(-1)^n} / X_n - X_n / \Lambda V_n^{(-1)^n}) = [Z_N K_N / i^p]^{en} \prod_{a=1}^{[N]} (\Lambda / Z_a - Z_a / \Lambda), \quad (3.57)$$

from which the fact that $J(X; K, U, V) \neq 0$ trivially follows. \square

We conclude that our construction of $B(\lambda)$ -eigenstates will work if the representations $\mathcal{R}_N, \mathcal{R}_M$ and \mathcal{R}_{N-M} are all non-degenerate. Theorem 1 follows by induction.

4. Characterization of T -spectrum

4.1 SOV characterization of T -spectrum

In the SOV representations the spectral problem for $T(\lambda)$ is reduced to the following discrete system of Baxter-like equations in the wave-function $\Psi_t(\eta) \equiv \langle \eta | t \rangle$ of a T -eigenstate $|t\rangle$:

$$t(\eta_r) \Psi(\eta) = a(\eta_r) \Psi(T_r^-(\eta)) + d(\eta_r) \Psi(T_r^+(\eta)) \quad \forall r \in \{1, \dots, [N]\}, \quad (4.1)$$

where $(\eta_1, \dots, \eta_N) \in \mathbb{B}_N$ with \mathbb{B}_N the set of zeros of the B -operator in the SOV representation. Here we have denoted with $T_r^\pm(\eta) \equiv (\eta_1, \dots, q^{\pm 1} \eta_r, \dots, \eta_N)$ and $a(\eta_r)$ and $d(\eta_r)$ the coefficients of the SOV representation as defined in subsection 3.2.2. In the case of N even, we have

¹¹Let $\sigma_n^{[N]}(Z)$ be the degree n elementary symmetric polynomial in the variables Z_r , then $\sigma_n^{[N]}(Z)/\sigma_{[N]}^{[N]}(Z)$ are Laurent polynomials of maximal degree 1 in each one of the parameters K_n, X_n, U_n, V_n .

to add to the system (4.1) the following equation in the variable η_N :

$$\mathsf{T}_N^+ \Psi_k(\eta) = q^{-k} \Psi_k(\eta), \quad (4.2)$$

for $t(\lambda) \in \Sigma_{\mathsf{T}}^{\theta, k}$ with $k \in \{0, \dots, 2l\}$ and $\theta \equiv \eta_A \prod_{a=1}^{N-1} \eta_a / \prod_{a=1}^N \xi_a$, obtained from the asymptotics of $\mathsf{T}(\lambda)$ given in appendix B.

4.2 Characterization of T -eigenvalues as solutions of a functional equation

Let us introduce the one parameter family $D(\lambda)$ of $p \times p$ matrix:

$$D(\lambda) \equiv \begin{pmatrix} t(\lambda) & -d(\lambda) & 0 & \dots & 0 & -a(\lambda) \\ -a(q\lambda) & t(q\lambda) & -d(q\lambda) & 0 & \dots & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & \dots & & & \vdots \\ \vdots & & & \dots & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \dots & 0 & -a(q^{2l-1}\lambda) & t(q^{2l-1}\lambda) & -d(q^{2l-1}\lambda) \\ -d(q^{2l}\lambda) & 0 & \dots & 0 & -a(q^{2l}\lambda) & t(q^{2l}\lambda) \end{pmatrix} \quad (4.3)$$

where for now $t(\lambda)$ is just a real even Laurent polynomial of degree \bar{N} in λ .

Lemma 6. *The determinant of the matrix $D(\lambda)$ is an even Laurent polynomial of maximal degree \bar{N} in $\Lambda \equiv \lambda^p$.*

Proof. Let us start observing that $D(\lambda q)$ is obtained by $D(\lambda)$ exchanging the first and p -th column and after the first and p -th row, so that

$$\det_p D(\lambda q) = \det_p D(\lambda) \quad \forall \lambda \in \mathbb{C}, \quad (4.4)$$

Being $a(\lambda)$, $d(\lambda)$ and $t(\lambda)$ even Laurent polynomial of degree \bar{N} in λ then the parity of $\det_p D(\Lambda)$ and the fact that it is a Laurent polynomial of maximal degree \bar{N} trivially follow. \square

The interest toward the function $\det_p D(\Lambda)$ is due to the following:

Lemma 7. *Let $t(\lambda) \in \Sigma_{\mathsf{T}}$, then $t(\lambda)$ is a solution of the functional equation:*

$$\det_p D(\Lambda) \equiv 0. \quad (4.5)$$

Proof. Note that from $t(\lambda) \in \Sigma_T$ and the SOV characterization of the T-spectrum we have that the requirement that the system of equations (4.1) admits a non-zero solution reads:

$$\det_p D(\eta_a^p) = 0 \quad \forall a \in \{1, \dots, [N]\} \text{ and } (\eta_1, \dots, \eta_{[N]}) \in \mathbb{B}_N, \quad (4.6)$$

In the case of N even, we have moreover:

$$\lim_{\log \Lambda \rightarrow \mp \infty} \Lambda^{\pm N} \det_p D(\Lambda) = 0, \quad (4.7)$$

which simply follows by observing that:

$$\begin{aligned} \lim_{\log \Lambda \rightarrow \mp \infty} \Lambda^{\pm N} \det_p D(\Lambda) &= \det_p \left\| q^{(1 \mp 1)N} \theta^{\pm} \delta_{i,j-1} + q^{-(1 \mp 1)N} \theta^{\mp} \delta_{i,j+1} - (q^k \theta + q^{-k} \theta^{-1}) \delta_{i,j} \right\| \\ &\quad \times (-1) \prod_{n=1}^N (i \kappa_n \xi_n^{\pm})^p = 0. \end{aligned} \quad (4.8)$$

for $t(\lambda) \in \Sigma_T^{\theta,k}$ and $k \in \{0, \dots, 2l\}$.

Then the function $\det_p D(\Lambda)$ is zero in $N + e_N$ different values of Λ^2 which thanks to Lemma 6 implies the statement of the Lemma. \square

Remark 2. Let us note that the same kind of functional equation $\det D(\Lambda) = 0$ also appears in [48, 49, 50, 51]. There it recasts, for different integrable quantum models at the *roots of unit*, the functional relations which result from the truncated fusions of transfer matrix eigenvalues; this is in particular true for the τ_2 -model¹² [25, 51, 52, 53].

4.3 Construction of Baxter equation solutions from T-eigenvalues

Theorem 2. For any $t(\lambda) \in \Sigma_T$, we can construct uniquely up to normalization a real polynomial:

$$Q_t(\lambda) = \lambda^{a_t} \prod_{h=1}^{2l\bar{N}-b_t} (\lambda^2 - \lambda_h^2), \quad 0 \leq a_t \leq 2l, \quad 0 \leq b_t \leq 2l\bar{N}, \quad (4.9)$$

which is a solution of the Baxter functional equation:

$$t(\lambda) Q_t(\lambda) = a(\lambda) Q_t(\lambda q^{-1}) + d(\lambda) Q_t(\lambda q) \quad \forall \lambda \in \mathbb{C}, \quad (4.10)$$

free from p -strings. Moreover, for N even and $t(\lambda) \in \Sigma_T^{\theta,k}$ with $k \in \{0, \dots, 2l\}$, it holds:

$$a_t = k, \quad b_t = k \bmod p. \quad (4.11)$$

¹²The SOV representations of this model were analyzed in a series of works [54, 55, 56].

Proof. Construction: Let us notice that the condition $t(\lambda) \in \Sigma_T$ implies that the $p \times p$ matrix $D(\lambda)$ has rank $2l$ for any $\lambda \in \mathbb{C} \setminus \{0\}$. Let us denote with

$$C_{i,j}(\lambda) = (-1)^{i+j} \det_{2l} D_{i,j}(\lambda) \quad (4.12)$$

the (i, j) cofactor of the matrix $D(\lambda)$; then the matrix formed out of these cofactors has rank 1, i.e. all the vectors:

$$V_i(\lambda) \equiv (C_{i,1}(\lambda), C_{i,2}(\lambda), \dots, C_{i,2l+1}(\lambda))^T \in \mathbb{C}^p \quad \forall i \in \{1, \dots, 2l+1\} \quad (4.13)$$

are proportional:

$$V_i(\lambda)/C_{i,1}(\lambda) = V_j(\lambda)/C_{j,1}(\lambda) \quad \forall i, j \in \{1, \dots, 2l+1\}, \quad \forall \lambda \in \mathbb{C}. \quad (4.14)$$

The proportionality (4.14) of the eigenvectors $V_i(\lambda)$ implies:

$$C_{2,2}(\lambda)/C_{2,1}(\lambda) = C_{1,2}(\lambda)/C_{1,1}(\lambda) \quad (4.15)$$

which, by using the property (A.2), can be rewritten as:

$$C_{1,1}(\lambda q)/C_{1,2l+1}(\lambda q) = C_{1,2}(\lambda)/C_{1,1}(\lambda). \quad (4.16)$$

Moreover, the first element in the vectorial condition $D(\lambda)V_1(\lambda) = 0$ reads:

$$t(\lambda)C_{1,1}(\lambda) = a(\lambda)C_{1,2l+1}(\lambda) + d(\lambda)C_{1,2}(\lambda). \quad (4.17)$$

In appendix A, we have analyzed the properties of these cofactors and thanks to Lemma 9 and 10 we can introduce now the polynomials $C_{1,1}\bar{C}_{1,1}(\lambda), C_{1,2}^*\bar{C}_{1,2l+1}(\lambda)$ and $C_{1,2}\bar{C}_{1,2}(\lambda)$ defined by simplifying the common factors in $C_{1,1}(\lambda), C_{1,2l+1}(\lambda)$ and $C_{1,2}(\lambda)$, respectively. In particular, from Lemma 9 and 10 we have that $\bar{C}_{1,1}(\lambda)$ is an even polynomial of the form:

$$\bar{C}_{1,1}(\lambda) = \prod_{h=1}^{\bar{N}_{1,1}} (\lambda^2 - \lambda_h^2), \quad \bar{N}_{1,1} \leq 2l\bar{N}, \quad (4.18)$$

which furthermore satisfies the properties:

$$\text{Reality: } (\bar{C}_{1,1}(\lambda))^* \equiv \bar{C}_{1,1}(\lambda^*) \quad \text{and} \quad 0 \notin Z_{\bar{C}_{1,1}(\lambda)}, \quad s_{p,\lambda_0} \notin Z_{\bar{C}_{1,1}(\lambda)}, \quad \forall \lambda_0 \in \mathbb{C}. \quad (4.19)$$

Now, being by definition $Z_{\bar{C}_{1,1}(\lambda)} \cap Z_{\bar{C}_{1,2}(\lambda)} = Z_{\bar{C}_{1,1}(\lambda)} \cap Z_{\bar{C}_{1,2l+1}(\lambda)} = \emptyset$, equation (4.16) implies:

$$\bar{C}_{1,2l+1}(\lambda) = q^{2\bar{N}_{1,1}} \bar{C}_{1,1}(\lambda q^{-1}), \quad \bar{C}_{1,2}(\lambda) = q^{-2\bar{N}_{1,1}} \bar{C}_{1,1}(\lambda q), \quad \varphi \equiv C_{1,1}/C_{1,2} \text{ a phase}, \quad (4.20)$$

and then equation (4.17) assumes the form of a Baxter equation in the polynomial $\bar{C}_{1,1}(\lambda)$:

$$t(\lambda)\bar{C}_{1,1}(\lambda) = \bar{a}(\lambda)\bar{C}_{1,1}(\lambda q^{-1}) + \bar{d}(\lambda)\bar{C}_{1,1}(\lambda q), \quad (4.21)$$

with coefficients $\bar{a}(\lambda) \equiv (q^{2\bar{N}_{1,1}}\varphi)a(\lambda)$ and $\bar{d}(\lambda) \equiv (q^{2\bar{N}_{1,1}}\varphi)^{-1}d(\lambda)$.

Let us denote with $\bar{D}(\Lambda)$ the matrix defined as in (4.3) but with coefficients $\bar{a}(\lambda)$ and $\bar{d}(\lambda)$. Then the consistence condition $\det_p \bar{D}(\Lambda) \equiv 0$ of the Baxter equation (4.21) and the condition $t(\lambda) \in \Sigma_T$ implies:

$$\det_p \bar{D}(\Lambda) - \det_p D(\Lambda) \equiv 0 \quad \longleftrightarrow \quad (\varphi^p - 1) (\mathcal{A}(\Lambda) - \varphi^{-p} \mathcal{D}(\Lambda)) \equiv 0, \quad (4.22)$$

which is true if and only if φ is a p -root of the unity. Note that (4.22) is derived by using the following expansion for $\det_p D(\Lambda)$ and a similar one for $\det_p \bar{D}(\Lambda)$:

$$\begin{aligned} \det_p D(\Lambda) &= \mathcal{A}(\Lambda) + \mathcal{D}(\Lambda) - a(\lambda)d(\lambda/q) \det_{2l-1} D_{(1,2l+1),(1,2l+1)}(\lambda) \\ &\quad - a(\lambda q)d(\lambda) \det_{2l-1} D_{(1,2),(1,2)}(\lambda) + t(\lambda) \det_{2l} D_{1,1}(\lambda), \end{aligned} \quad (4.23)$$

where $D_{(h,k),(h,k)}(\lambda)$ denotes the $(2l-1) \times (2l-1)$ sub-matrix of $D(\lambda)$ obtained removing the rows and columns h and k , plus the formulae¹³:

$$\det_{2l} \bar{D}_{1,1}(\lambda) = \det_{2l} D_{1,1}(\lambda), \quad (4.24)$$

$$\det_{2l-1} \bar{D}_{(1,2),(1,2)}(\lambda) = \det_{2l-1} D_{(1,2),(1,2)}(\lambda), \quad (4.25)$$

$$\det_{2l-1} \bar{D}_{(1,2l+1),(1,2l+1)}(\lambda) = \det_{2l-1} D_{(1,2l+1),(1,2l+1)}(\lambda). \quad (4.26)$$

Finally, we can define our polynomial solution (4.9) of the Baxter equation (4.10) by:

$$Q_t(\lambda) \equiv \lambda^{a_t} \bar{C}_{1,1}(\lambda), \quad (4.27)$$

where $q^{-a_t} \equiv q^{2\bar{N}_{1,1}}\varphi$ with $a_t \in \{0, \dots, 2l\}$ and $b_t \equiv 2l\bar{N} - \bar{N}_{1,1}$.

In the case N even, for $t(\lambda) \in \Sigma_T^{\theta,k}$ with $k \in \{0, \dots, 2l\}$, we get from the Baxter equation and the asymptotics of the coefficients (3.30) the following asymptotic conditions:

$$\lim_{\lambda \rightarrow 0} \frac{Q_t(\lambda q)}{Q_t(\lambda)} = q^k, \quad \lim_{\lambda \rightarrow \infty} \frac{Q_t(\lambda q)}{Q_t(\lambda)} = q^{-(2N+k)}. \quad (4.28)$$

Then the characterization (4.11) is just a corollary of (4.28). \square

Proof. Uniqueness: Let $t(\lambda) \in \Sigma_T$ and let $Q_t(\lambda)$ be the solution of the Baxter equation (4.10) constructed in the previous part of the proof. Then denoting with $\bar{Q}_t(\lambda) \in \mathbb{C}[\lambda]$ any other solution, we can define the q-Wronskian:

$$W_t(\lambda) = Q_t(\lambda)\bar{Q}_t(q^{-1}\lambda) - \bar{Q}_t(\lambda)Q_t(q^{-1}\lambda). \quad (4.29)$$

¹³They follow from Lemma 3 of [2] thanks to the *tridiagonality* of these matrices.

which by the Baxter equation satisfies the equation:

$$a(\lambda) W_t(\lambda) = d(\lambda) T^+ W_t(\lambda). \quad (4.30)$$

Thanks to the cyclicity $(T^+)^p = 1$ the average of the above equation reads:

$$(\mathcal{A}(\Lambda) - \mathcal{D}(\Lambda)) \mathcal{W}_t(\Lambda) = 0 \quad \text{with} \quad \mathcal{W}_t(\Lambda) \equiv \prod_{k=0}^{2l} W_t(\lambda q^k), \quad (4.31)$$

which by Lemma 5 implies $W(\lambda) \equiv 0$. It is then easy to see that this implies that $\bar{Q}_t(\lambda) \equiv Q_t(\lambda)$ up to normalization when we have assumed $\bar{Q}_t(\lambda)$ free of p -strings. \square

Remark 3. The previous theorem implies that for any $t(\lambda) \in \Sigma_T$ the polynomial solution $Q_t(\lambda)$ of the Baxter equation can be related to the determinant of a tridiagonal matrix of finite size $p - 1$ for the rational $\beta^2 = p'/p$. It is then relevant to remark that others determinant representations for Baxter equation solutions are known in literature; for example in the quantum periodic Toda chain. There, these solutions are expressed in terms of linear combinations of determinants of semi-infinite tridiagonal matrices [26, 57, 58]. The above remark explains the interest in a careful analysis for the SG model of the limit $\beta^2 \rightarrow \bar{\beta}^2$ with $\bar{\beta}^2$ irrational. Indeed, under this limit (i.e. $p', p \rightarrow +\infty$) the dimension of the representation in any quantum site diverges as well as the size of the tridiagonal matrix associated to a solution $Q_t(\lambda)$ of the Baxter equation. Then, it is natural to investigate if characterizations like those encountered for the quantum periodic Toda chain apply to the SG model for irrational $\bar{\beta}^2$, too. In particular, this is interesting for a potential reformulation¹⁴ of the SG spectrum in terms of solutions of nonlinear integral equations (NLIE) as shown recently in [61] for the quantum Toda chain. On the other hand, it is worth remarking that the SG spectrum admits another NLIE reformulation which is of DDV-type¹⁵ and satisfies the feature of completeness as consequence of the Bethe ansatz completeness proven in Subsection 4.6.

4.4 Construction of T-eigenstates from T-eigenvalues

The results of the previous subsections allow both to introduce a complete characterization of the set Σ_T and to construct one T-eigenstate $|t\rangle$ for any $t(\lambda) \in \Sigma_T$; indeed, we can prove the following:

¹⁴The use of NLIE to reformulate the spectrum characterization in integrable quantum models goes back to [59, 60]. NLIE reformulations similar to that presented in [61] also appear in [18, 62, 45, 63].

¹⁵ This type of NLIE are derived by a reformulation of the Bethe ansatz equations and they were introduced and analyzed for fermionic lattice regularizations of the Sine-Gordon model in [64]-[71] and in [72]-[75] for a related model.

Theorem 3. *The set Σ_T coincides with the set of all the $t(\lambda) \in \mathbb{R}[\lambda^2, \lambda^{-2}]_{\bar{N}/2}$ solutions of the functional equation:*

$$\det_p D(\Lambda) = 0, \quad \forall \Lambda \in \mathbb{C}. \quad (4.32)$$

Moreover, for any $t(\lambda) \in \Sigma_T$ a corresponding T -eigenstate is characterized by:

$$\Psi_t(\eta) \equiv \langle \eta_1, \dots, \eta_N | t \rangle = \prod_{r=1}^N Q_t(\eta_r), \quad \text{if } N \text{ is odd}, \quad (4.33)$$

$$\Psi_t(\eta) \equiv \langle \eta_1, \dots, \eta_N | t \rangle = \eta_N^{-k} \prod_{r=1}^{N-1} Q_t(\eta_r), \quad \text{if } N \text{ is even and } t(\lambda) \in \Sigma_T^{\theta, k}, \quad (4.34)$$

where $Q_t(\lambda)$ is the Baxter equation solution constructed in Theorem 2.

Proof. The inclusion of Σ_T in that set of solutions of (4.32) was proven in Lemma 7. To prove the reverse inclusion of sets, we have just to observe that the construction of $Q_t(\lambda)$, presented in Theorem 2, holds for any $t(\lambda) \in \mathbb{R}[\lambda^2, \lambda^{-2}]_{\bar{N}/2}$ solution of (4.32). Then, the $t(\lambda)$ and the $\Psi_t(\eta)$, (4.33) and (4.34), are solutions of the system of Baxter like equations (4.1) and (4.2) and so they define a T -eigenvalue and a corresponding T -eigenstate. \square

4.5 Simplicity of T -spectrum

In this section we show that the spectrum of the transfer matrix $T(\lambda)$ is non-degenerate (or simple). Let us start proving the following:

Lemma 8. *For almost all the values of the parameters K_n, X_n, U_n, V_n , the average values of the monodromy matrix elements $A(\lambda)$ and $D(\lambda)$ satisfy the inequalities:*

$$A(Z_a) \neq D(Z_a), \quad \forall a \in \{1, \dots, [N]\}, \quad (4.35)$$

where Z_a are the zeros of the average value of $B(\lambda)$.

Proof. We have to prove that the functions:

$$\mathcal{F}_a(K_n, X_n, U_n, V_n) \equiv \mathcal{A}_N(Z_a) - \mathcal{D}_N(Z_a), \quad \forall a \in \{1, \dots, [N]\} \quad (4.36)$$

are nonzero for almost all the values of the parameters K_n, X_n, U_n, V_n . The Lemma 4 and the functional dependence¹⁶ of the $Z_1, \dots, Z_{[N]}$ w.r.t. these parameters implies that it is sufficient to show that the functions \mathcal{F}_a are nonzero for some special value of the parameters in order to prove our Lemma. Note that the following identities hold:

$$\mathcal{F}_a |_{U_n=1 \forall n \in \{2, \dots, N\}, V_n=1 \forall n \in \{1, \dots, N\}} = (1 + K_1^2)(U_1 - U_1^{-1})\mathcal{A}_{2, N-1}(Z_a), \quad (4.37)$$

¹⁶See footnote 11.

Here, we have used the decomposition of the chain in a first subchain $\mathbf{1}$, formed by the site 1, and a second subchain $\mathbf{2}$, formed by the remaining sites. Moreover, we have used the identities:

$$(\mathcal{A}_{\mathbf{1},1} - \mathcal{D}_{\mathbf{1},1}) = (1 + K_1^2)(U_1 - U_1^{-1}), \quad \mathcal{B}_{\mathbf{1},1}(\Lambda) = \mathcal{C}_{\mathbf{1},1}(\Lambda), \quad (4.38)$$

$$\mathcal{B}_{\mathbf{2},N-1}(\Lambda) = \mathcal{C}_{\mathbf{2},N-1}(\Lambda), \quad \mathcal{D}_{\mathbf{2},N-1}(\Lambda) = \mathcal{A}_{\mathbf{2},N-1}(\Lambda), \quad (4.39)$$

which follows by using Lemma 3 and (3.21) for the special choice of the parameters done in (4.37). Then for $K_1 \neq \pm i$ and $U_1 \neq \pm 1$, we have only to show that we can always take $\mathcal{A}_{\mathbf{2},N-1}(Z_a) \neq 0$ for any $a \in \{1, \dots, [N]\}$.

Note that from the identities:

$$\det_q \mathcal{M}_{\mathbf{2},N-1}(\Lambda) = (\mathcal{A}_{\mathbf{2},N-1}(\Lambda))^2 - (\mathcal{B}_{\mathbf{2},N-1}(\Lambda))^2, \quad (4.40)$$

$$\mathcal{B}_N(\Lambda) = \mathcal{A}_{\mathbf{1},1} \mathcal{B}_{\mathbf{2},N-1}(\Lambda) + \mathcal{B}_{\mathbf{1},1}(\Lambda) \mathcal{A}_{\mathbf{2},N-1}(\Lambda), \quad (4.41)$$

we have that $\mathcal{A}_{\mathbf{2},N-1}(Z_a) = 0$ if and only if Z_a is a double zero of $\det_q \mathcal{M}_{\mathbf{2},N-1}(\Lambda)$. However, this is not the case in general because the function $\det_q \mathcal{M}_{\mathbf{2},N-1}(\Lambda)$ has not double zeros for general values of the parameters X_n and K_n ; as it is clear from the formula:

$$\det_q \mathcal{M}_{\mathbf{2},N-1}(\Lambda) \equiv \prod_{h=\pm 1} \prod_{n=2}^N K_n^2(\Lambda/M_{n,h} - M_{n,h}/\Lambda), \quad M_{n,\pm} \equiv \pm i^p K_n^{\pm 1} X_n. \quad (4.42)$$

which is obtained averaging the quantum determinant (3.6). \square

Theorem 4. *For almost all the values of the parameters K_n, X_n, U_n, V_n of a twisted representation, the spectrum of $\mathsf{T}(\lambda)$ is simple.*

Proof. We have to prove that up to normalization for any given $t(\lambda) \in \Sigma_{\mathsf{T}}$ the wave-function (4.33) and (4.34) are the only solutions of the system (4.1) and (4.2).

Let us denote with $\bar{\Psi}_t(\eta)$, $\eta \in \mathbb{B}_N$, any other solution corresponding to the same T -eigenvalue $t(\lambda)$. Then, we can define the q-Wronskian:

$$W_{t,r}(\eta) = \Psi_t(\eta) \bar{\Psi}_t(\mathsf{T}_r^-(\eta)) - \bar{\Psi}_t(\eta) \Psi_t(\mathsf{T}_r^-(\eta)), \quad \forall r \in \{1, \dots, [N]\}. \quad (4.43)$$

which by the system of Baxter like equations (4.1) satisfy the equations:

$$\mathbf{a}(\eta_r) W_{t,r}(\eta) = \mathbf{d}(\eta_r) W_{t,r}(\mathsf{T}_r^+(\eta)), \quad \forall r \in \{1, \dots, [N]\}. \quad (4.44)$$

Thanks to the cyclicity $(\mathsf{T}_r^+)^p = 1$, the averages of the above equations read:

$$(\mathcal{A}(Z_r) - \mathcal{D}(Z_r)) \mathcal{W}_{t,r}(\eta) = 0 \quad \text{with} \quad \mathcal{W}_{t,r}(\eta) \equiv \prod_{k=0}^{2l} W_{t,r}(\mathsf{T}_r^k(\eta)), \quad (4.45)$$

which by Lemma 8 implies:

$$W_{t,r}(\eta) = 0 \leftrightarrow \frac{\bar{\Psi}_t(\mathbf{T}_r^+(\eta))}{\bar{\Psi}_t(\eta)} = \frac{Q_t(\eta_r q)}{Q_t(\eta_r)}, \quad \forall \eta \in \mathbb{B}_N \text{ and } \forall r \in \{1, \dots, [N]\}. \quad (4.46)$$

In the case N odd, these identities just imply the uniqueness of the solution of the system (4.1) and its factorized form given in (4.33). In the case N even, they imply the factorization:

$$\bar{\Psi}_t(\eta) = \bar{f}_t(\eta_N) \prod_{r=1}^{N-1} Q_t(\eta_r). \quad (4.47)$$

Then we get $\bar{f}_t(\eta_N) \propto \eta_N^{-k}$ as the unique solution of the equation (4.2) in η_N for $t(\lambda) \in \Sigma_{\mathbf{T}}^{\theta,k}$. \square

Remark 4. It is worth to point out that for the twisted representations of the SG model the spectrum of the transfer matrix is simple both for N odd and even. This is due to the fact that by definition for N even these representations are characterized by an average value of the Θ -charge different from 1, i.e. $\theta^p \neq 1$.

4.6 Completeness of the Bethe ansatz

An important consequence of the previous analysis is that it naturally leads to the complete characterization of the transfer matrix spectrum (eigenvalue and eigensate) in terms of real polynomial solutions of the associated Baxter functional equation.

Let us observe that Theorem 2 implies that to any $t(\lambda) \in \Sigma_{\mathbf{T}}$ it is associated one and only one self-complex-conjugate solution¹⁷ $(\pm\lambda_1, \dots, \pm\lambda_{2l\bar{N}-b})$ of the following system of Bethe equations:

$$\frac{a(\pm\lambda_k)}{d(\pm\lambda_k)} = -q^{2a} \prod_{h=1}^{2l\bar{N}-b} \frac{(q^2\lambda_k^2 - \lambda_h^2)}{(\lambda_k^2/q^2 - \lambda_h^2)}, \quad \forall k \in \{1, \dots, 2l\bar{N} - b\}. \quad (4.48)$$

Conversely, let us consider a self-complex-conjugate solution of the Bethe system of equations (4.48), then it defines uniquely a real polynomial $Q(\lambda)$ by the equation (4.9). Now, by using $Q(\lambda)$ we can construct the function:

$$t(\lambda) = (a(\lambda)Q(q^{-1}\lambda) + d(\lambda)Q(q\lambda))/Q(\lambda). \quad (4.49)$$

which, thanks to the Bethe equations (4.48), is nonsingular for $\lambda = \pm\lambda_k$, $\forall k \in \{1, \dots, 2l\bar{N}-b\}$ and is a real even Laurent polynomial of degree \bar{N} in λ . Moreover, we can also uniquely construct a state $|t\rangle$ by inserting $Q(\lambda)$ in equation (4.33) for N odd (and (4.34) for N even). Then, $t(\lambda)$ and the wave-function $\Psi_t(\eta) \equiv \langle \eta_1, \dots, \eta_N | t \rangle$ satisfies by definition the system of Baxter like equations (4.1) (and (4.2) for N even). This implies that starting from the given

¹⁷It means that the tuple $(\pm\lambda_1, \dots, \pm\lambda_{2l\bar{N}-b})$ is invariant up to reordering under complex conjugation.

solution of the Bethe equation we have uniquely reconstructed the T -eigenvalue $t(\lambda)$ and the corresponding T -eigenstate $|t\rangle$. This establishes a one-to-one correspondence between the above solutions to (4.48) and the spectrum of the transfer matrix (*Completeness of the Bethe ansatz*).

4.7 Reconstruction of Q -operator

In the previous sections, we have given a complete and operative characterization of the spectrum of the transfer matrix $T(\lambda)$ starting from its SOV characterization. It is worth pointing out that we have derived our results without any need to introduce a Q -operator. Anyhow, it is worth noticing that our construction has as built-in property the existence and characterization of the Q -operator. This feature is even more relevant in the case of twisted representations of the SG model where a direct construction by cyclic dilogarithm functions can be applied only on the sub-variety of representations generated by points on algebraic curves.

Definition 1. Let $Q(\lambda)$ be the operator family defined by¹⁸:

$$Q(\lambda)|t\rangle \equiv \bar{Q}_t(\lambda)|t\rangle, \quad \text{where} \quad \bar{Q}_t(\lambda) \equiv \lambda^{p(1-e_{at})} Q_t(\lambda), \quad \forall t(\lambda) \in \Sigma_T. \quad (4.50)$$

Here, $Q_t(\lambda)$ is the real polynomial corresponding to $t(\lambda)$ by the injection defined in Theorem 2 and $|t\rangle$ is the corresponding T -eigenstate.

Then the following theorem holds:

Theorem 5. The operator family $Q(\lambda)$ is a Baxter Q -operator:

(A) $Q(\lambda)$ and $T(\lambda)$ satisfy the commutation relations:

$$[Q(\lambda), T(\mu)] = [Q(\lambda), Q(\mu)] = 0 \quad \forall \lambda, \mu \in \mathbb{C}, \quad (4.51)$$

plus the Baxter equation:

$$T(\lambda)Q(\lambda) = a(\lambda)Q(\lambda q^{-1}) + d(\lambda)Q(\lambda q) \quad \forall \lambda \in \mathbb{C}, \quad (4.52)$$

and for N even:

$$[Q(\lambda), \Theta] = 0 \quad \forall \lambda \in \mathbb{C}. \quad (4.53)$$

(B) $Q(\lambda)$ is a polynomial of degree $2l(\bar{N} + 1)$ in λ^2 :

$$Q(\lambda) \equiv \sum_{n=0}^{2l(\bar{N}+1)} Q_n \lambda^{2n}, \quad (4.54)$$

with coefficients Q_n self-adjoint operators.

¹⁸Here, we have decided to define $\bar{Q}_t(\lambda)$ by multiplying the function $Q_t(\lambda)$ for the p -string λ^p when a_t is odd. This is done to get that $\bar{Q}_t(\lambda)$ is always a real **even** polynomial in λ which is still a solution of (4.10).

Proof. Note that the self-adjointness of the transfer matrix $T(\lambda)$ implies that $Q(\lambda)$ is well defined being its action defined on a basis. The commutation relations and the Baxter equation are trivial consequences of Definition 1, thanks to the simplicity of the T -spectrum. Finally, the reality condition of the Q -eigenvalues:

$$(\bar{Q}_t(\lambda))^* = \bar{Q}_t(\lambda^*) \quad \forall \lambda \in \mathbb{C} \quad (4.55)$$

implies that the operators Q_n are self-adjoint. \square

Remark 5. In [53], Baxter has extended the construction of the Q -operator by gauge transformations of [25] to the τ_2 -model for general cyclic representations not restricted to those parameterized by points on algebraic curves. The main tool there was a generalization of the discrete dilogarithm functions. In particular, he has remarked that asking the cyclicity only for the products of couples of these functions the Q -operator can be still well defined for general cyclic representations of the model. Note that there are no doubts that this construction can be adapted to our general representations of the SG model leading to the representation of the Q -operator in the basis (2.5) of the representation. However, it is also clear that this last characterization of Q cannot add any information to our construction of the transfer matrix spectrum but it can be potentially useful for more *physical aims*. In particular, we can try to use it to investigate the lattice dynamics [76, 77] associated to the general cyclic representations of the SG model. This can be done following the the same line drawn in [1], i.e. showing that the discrete evolution operator can be written in terms of the Q -operator. Here, the main *physical aim* is to prove that these representations of the SG model define lattice regularizations of the Sine-Gordon model for general α -sectors of the quantum theory (see Section 2.2 of [1] for the definition).

A. Properties of the cofactors $C_{i,j}(\lambda)$

Let $t(\lambda)$ be just a real even Laurent polynomial of degree \bar{N} in λ , then the cofactors of the matrix $D(\lambda)$ satisfy the following:

Lemma 9. *The cofactors $C_{i,j}(\lambda)$ are even Laurent polynomials of maximal degree¹⁹ $2l\bar{N}$:*

$$C_{i,j}(\lambda) = C_{i,j} \lambda^{-2l\bar{N}+2a_{i,j}} \prod_{h=1}^{2l\bar{N}-(a_{i,j}+b_{i,j})} (\lambda^2 - \lambda_{(i,j),h}^2), \quad (A.1)$$

which satisfy the following properties:

$$C_{h+i,k+i}(\lambda) = C_{h,k}(\lambda q^i) \quad \forall i, h, k \in \{1, \dots, p\}. \quad (A.2)$$

¹⁹The $a_{i,j}$ and $b_{i,j}$ are nonnegative integers and by definition $\lambda_{(i,j),h} \neq 0$ for any $h \in \{1, \dots, 2l\bar{N} - (a_{i,j} + b_{i,j})\}$.

and

$$(\mathbf{C}_{1,1}(\lambda))^* \equiv \mathbf{C}_{1,1}(\lambda^*), \quad (\mathbf{C}_{1,2}(\lambda))^* \equiv \mathbf{C}_{1,2l+1}(\lambda^*). \quad (\text{A.3})$$

Proof. The characterization (A.1) of the cofactors trivially follows from being $\mathbf{a}(\lambda)$, $\mathbf{d}(\lambda)$ and $t(\lambda)$ even Laurent polynomial of degree \bar{N} in λ . Note that by the definition (4.12) of the cofactors $\mathbf{C}_{i,j}(\lambda)$ the equations (A.2) are simple consequences of $q^p = 1$ and are proven exchanging rows and columns in the determinants.

Now let us prove the property (A.3) for the cofactor $\mathbf{C}_{1,1}(\lambda) = \det_{2l} D_{1,1}(\lambda)$, where:

$$D_{1,1}(\lambda) \equiv \left\| t(\lambda q^h) \delta_{h,k} - \mathbf{a}(\lambda q^h) \delta_{h,k+1} - \mathbf{d}(\lambda q^h) \delta_{h,k-1} \right\|_{1 \leq h \leq 2l, 1 \leq k \leq 2l}, \quad (\text{A.4})$$

then by the properties under complex conjugation of $t(\lambda)$, $\mathbf{a}(\lambda)$ and $\mathbf{d}(\lambda)$ it holds:

$$(D_{1,1}(\lambda))^* \equiv \left\| t(\lambda^* q^h) \delta_{h,k} - \mathbf{d}(\lambda^* q^{p-h}) \delta_{h,k+1} - \mathbf{a}(\lambda^* q^{p-h}) \delta_{h,k-1} \right\|_{1 \leq h \leq 2l, 1 \leq k \leq 2l}, \quad (\text{A.5})$$

Let us denote with $D_{1,1}^{\mathcal{C}}$ the $2l \times 2l$ matrix of columns:

$$C_a^{D_{1,1}^{\mathcal{C}}} \equiv C_{p-a}^{D_{1,1}}, \quad \forall a \in \{1, \dots, 2l\}, \quad (\text{A.6})$$

where C_a^X stays for the column a of the matrix X , and similarly with $D_{1,1}^{\mathcal{C},\mathcal{R}}$ the $2l \times 2l$ matrix of rows:

$$R_a^{D_{1,1}^{\mathcal{C},\mathcal{R}}} \equiv R_{p-a}^{D_{1,1}^{\mathcal{C}}}, \quad \forall a \in \{1, \dots, 2l\}, \quad (\text{A.7})$$

where R_a^X stays for the row a of the matrix X . Then we get the identity:

$$D_{1,1}^{\mathcal{C},\mathcal{R}}(\lambda^*) \equiv (D_{1,1}(\lambda))^* \rightarrow (\mathbf{C}_{1,1}(\lambda))^* \equiv \mathbf{C}_{1,1}(\lambda^*), \quad (\text{A.8})$$

being the determinant invariant under an even number of column and row exchanges. In a completely similar way, we can prove the identity:

$$(\det_{2l-1} D_{(1,2),(1,2)}(\lambda))^* \equiv \det_{2l-1} D_{(1,2),(1,2)}(\lambda^*/q), \quad (\text{A.9})$$

then by using the cofactor expansions:

$$\mathbf{C}_{1,2l+1}(\lambda) = \prod_{h=1}^{2l} \mathbf{a}(\lambda q^h) + \mathbf{d}(\lambda/q) \det_{2l-1} D_{(1,2),(1,2)}(\lambda/q), \quad (\text{A.10})$$

$$\mathbf{C}_{1,2}(\lambda) = \prod_{h=1}^{2l} \mathbf{d}(\lambda q^h) + \mathbf{a}(\lambda q) \det_{2l-1} D_{(1,2),(1,2)}(\lambda). \quad (\text{A.11})$$

the second identity in (A.3) is a corollary of (A.9). \square

Lemma 10. *Let $t(\lambda) \in \Sigma_T$, then the following identities hold:*

$$a_{1,1} = a_{1,2}, \quad b_{1,1} = b_{1,2} \quad (\text{A.12})$$

where $a_{i,j}$ and $b_{i,j}$ are the nonnegative integers defined in (A.1),

$$Z_{C_{1,1}(\lambda)} \cap Z_{C_{1,2}(\lambda)} \equiv Z_{C_{1,1}(\lambda)} \cap Z_{C_{1,2l+1}(\lambda)}, \quad (\text{A.13})$$

and if

$$\exists s_{p,\lambda_0} \subset Z_{C_{1,1}(\lambda)} \rightarrow s_{p,\lambda_0} \cap Z_{C_{1,2}(\lambda)} \neq \emptyset, \quad (\text{A.14})$$

for $s_{p,\lambda_0} \equiv (\lambda_0, q\lambda_0, \dots, q^{2l}\lambda_0)$ any p -string.

Proof. Under the assumption $t(\lambda) \in \Sigma_T$ we have proven the validity of the equations (4.16) and (4.17). Then the identity (A.12) follows by using (4.16) and the property (A.3). Concerning the proof of identity (A.13) thanks to the prescriptions (3.23)-(3.25), chosen in subsection 3.2.2, we can follow word by word the proof given in Lemma 5 of the paper [2].

Let us assume by absurd that (A.14) is not satisfied, then (4.16) implies that $s_{p,\lambda_0} \subset Z_{C_{1,2l+1}(\lambda)}$. Now this last condition implies that (4.17) can be satisfied only if $s_{p,\lambda_0} \subset Z_{d(\lambda)}$, which does not hold by the definition of $d(\lambda)$. \square

B. Asymptotics of Yang-Baxter generators

From the known form of the Lax operator we derive the following asymptotics for $\lambda \rightarrow +\infty$ and 0 of the generators of the Yang-Baxter algebras.

N odd: The leading operators are $B_N(\lambda)$ and $C_N(\lambda)$ with asymptotics:

$$B_N(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a}{i} \right) \left(\lambda^N \prod_{a=1}^N \frac{v_a^{(-1)^{1+a}}}{\xi_a} - \lambda^{-N} \prod_{a=1}^N \xi_a v_a^{(-1)^a} \right) + \text{sub-leading}, \quad (\text{B.1})$$

$$C_N(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a}{i} \right) \left(\lambda^N \prod_{a=1}^N \frac{v_a^{(-1)^a}}{\xi_a} - \lambda^{-N} \prod_{a=1}^N \xi_a v_a^{(-1)^{1+a}} \right) + \text{sub-leading}. \quad (\text{B.2})$$

N even: The leading operators are $A_N(\lambda)$ and $D_N(\lambda)$ with asymptotics:

$$A_N(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a}{i} \right) \left(\lambda^N \prod_{a=1}^N \frac{v_a^{(-1)^a}}{\xi_a} + \lambda^{-N} \prod_{a=1}^N \xi_a v_a^{(-1)^{1+a}} \right) + \text{sub-leading}, \quad (\text{B.3})$$

$$D_N(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a}{i} \right) \left(\lambda^N \prod_{a=1}^N \frac{v_a^{(-1)^{1+a}}}{\xi_a} + \lambda^{-N} \prod_{a=1}^N \xi_a v_a^{(-1)^a} \right) + \text{sub-leading}. \quad (\text{B.4})$$

Note that these asymptotics imply for the SOV representation of the Yang-Baxter generators the following formulae²⁰:

N odd:

$$(w^{\text{SOV}})^{-1} \left(\prod_{a=1}^N v_a^{(-1)^{1+a}} \right) w^{\text{SOV}} = \prod_{a=1}^N \frac{\xi_a}{\eta_a}. \quad (\text{B.5})$$

N even:

$$\prod_{a=1}^N \xi_a (w^{\text{SOV}})^{-1} \Theta w^{\text{SOV}} = \left(\eta_A \prod_{a=1}^{N-1} \eta_a \right) \mathbb{T}_N^-, \quad (\text{B.6})$$

$$\prod_{a=1}^N \xi_a (w^{\text{SOV}})^{-1} \Theta^{-1} w^{\text{SOV}} = \left(\eta_D \prod_{a=1}^{N-1} \eta_a \right) \mathbb{T}_N^+, \quad (\text{B.7})$$

Note that taking the average value of the last two formulae we get for N odd:

$$\prod_{a=1}^N \frac{X_a}{Z_a} = \prod_{a=1}^N V_a^{(-1)^{1+a}}, \quad (\text{B.8})$$

while for N even:

$$Z_A = \langle \Theta \rangle \prod_{a=1}^{N-1} Z_a^{-1} \prod_{a=1}^N X_a, \quad Z_D = Z_A \langle \Theta \rangle^{-2}, \quad (\text{B.9})$$

where

$$\langle \Theta \rangle = \prod_{a=1}^N V_a^{(-1)^{1+a}}, \quad (\text{B.10})$$

is the average value of the charge Θ .

C. Comparison between the τ_2 -model and the SG model

In this appendix we present the comparison between the SG model and the so-called τ_2 -model introduced and analyzed in the series of papers [25, 51]-[56].

C.1 Lax operators

The Lax operator which describes the τ_2 -model reads:

$$L_n^{\tau_2} \equiv A_n \begin{pmatrix} D_n \equiv (d_{n,+} X_n + d_{n,-} X_n^{-1}) & H_n \equiv (h_{n,-} X_n + h_{n,+} X_n^{-1}) Y_n \\ G_n \equiv (g_{n,-} X_n + g_{n,+} X_n^{-1}) Y_n^{-1} & F_n \equiv (f_{n,-} X_n + f_{n,+} X_n^{-1}) \end{pmatrix} A_n^{-1}, \quad (\text{C.1})$$

²⁰Here, we denote with w^{SOV} the invertible matrix which defines the change of basis in the representation from the basis obtained by the tensor product of the v_n -eigenbasis to the B-eigenbasis.

where:

$$A_n \equiv \begin{pmatrix} u_n & 0 \\ 0 & 1 \end{pmatrix} \quad (C.2)$$

and:

$$g_{n,+} = f_{n,+}d_{n,-}/h_{n,+}, \quad g_{n,-} = f_{n,-}d_{n,+}/h_{n,-}. \quad (C.3)$$

Here, we have denoted with $Y_n \equiv u_n/u_n$ and $X_n \equiv v_n/v_n$ the generators of the local Weyl algebra $Y_n X_n = q X_n Y_n$ with the following central values $X_n^p = Y_n^p = 1$.

Then, the Lax operator of the τ_2 -model can be related to one of the SG model by:

$$L_n^{\tau_2} = L_n^{SG} \sigma_1, \quad (C.4)$$

when the parameters in (C.1) are identified by:

$$d_{n,-} = d_{n,+}^{-1}, \quad f_{n,-} = f_{n,+}^{-1}, \quad (C.5)$$

$$f_{n,+} = -i\lambda_n v_n^{-1}, \quad d_{n,+} = -i\lambda_n v_n, \quad (C.6)$$

$$h_{n,-} = q^{1/2} \kappa_n v_n, \quad h_{n,+} = q^{-1/2} \kappa_n^{-1} v_n^{-1}, \quad (C.7)$$

$$g_{n,+} = q^{1/2} \kappa_n v_n^{-1}, \quad g_{n,-} = q^{-1/2} \kappa_n^{-1} v_n. \quad (C.8)$$

It is worth pointing out that in our SG model, we are asking that the elements of the Lax operator generate the algebra $U_q(sl_2)$ and that such a requirement implies the relations (C.5).

It is worth pointing out the different role played by the parameters u_n in these two models. From (C.1), we see that the parameter u_n enters in the τ_2 -Lax operator as a simple gauge while this is not the case for the SG Lax operator being²¹ $[\sigma_1, A_n] \neq 0$. In particular, this means that while we are free to change uniformly along the chain the average values of the local generators u_n without any modification in the τ_2 -transfer matrix this is not the case for the SG transfer matrix.

C.2 Map from the SG model to the τ_2 -model for even chain

The relation (C.4) directly implies that for an odd chains the transfer matrices in the two models have different spectrum. In the case of even chain a more precise analysis is need and it is here developed.

Let us fix $N = 2M$ with M positive integer; then, it is simple to define a map from the generators of the Yang-Baxter algebras in the SG model to those in the τ_2 -model. We have just to point out that the relation (C.1) implies:

$$M^{\tau_2}(\lambda) = \sigma_1 \Pi_{\tau_2} (M^{SG}(\lambda)) \sigma_1 \longrightarrow T^{\tau_2}(\lambda) = \Pi_{\tau_2} (T^{SG}(\lambda)). \quad (C.9)$$

²¹The presence of the σ_1 tells us that what is a simple gauge for the τ_2 -model is not a gauge for the SG model and vice versa.

where we have defined:

$$\Pi_{\tau_2} (L_{2m-a}^{SG}(\lambda_{2m-a})) \equiv (\sigma_1)^{1-a} L_{2m-a}^{SG}((-1)^{(1-a)}\lambda_{2m-a}) (\sigma_1)^{1-a} = \begin{cases} L_{2m-1}^{SG}(\lambda_{2m-1}), \\ (L_{2m}^{SG}(-\varepsilon\lambda_{2m}^*))^\dagger, \end{cases} \quad (\text{C.10})$$

for $m \in \{1, \dots, M\}$ and $a \in \{0, 1\}$; note that in the last equality we have used (2.14). Then, we can characterize the map Π_{τ_2} by its action on the generators of the local Weyl algebras:

$$\Pi_{\tau_2} \equiv \prod_{n=1}^M F_n, \quad \text{where: } F_n(u_n) = u_n^{-1}, \quad F_n(v_n) = v_n^{-1}. \quad (\text{C.11})$$

Until now, the discussion presented is common for both the untwisted and twisted representations of the SG model. However, the distinction between these two cases appears evident when we observe that:

Untwisted representations: The map Π_{τ_2} is realized by the unitary transformation π_{τ_2} :

$$\Pi_{\tau_2}(M^{SG}(\lambda)) = \pi_{\tau_2} M^{SG}(\lambda) \pi_{\tau_2}^\dagger, \quad \pi_{\tau_2} \equiv \prod_{n=1}^M \Omega_{2n}, \quad (\text{C.12})$$

after the flipping $\xi_{2n-a} \rightarrow (-\varepsilon)^{(1-a)}\xi_{2n-a}$ of the inhomogeneities. Here, ε is the sign defined in (2.12) and Ω_n are the local unitary transformations defined by the following action:

$$\Omega_n |z_1, \dots, z_N\rangle \equiv |z_1, \dots, z_n^{-1}, \dots, z_N\rangle$$

on the elements of the basis (2.4) of the space of the representation.

Twisted representations: The map Π_{τ_2} **cannot** be realized by any similarity transformation on the space of the representation.

Indeed, the distinction between these two cases can be simply understood by the identities:

$$F_n(u_n) = (\Omega_n u_n \Omega_n) / u_n^2, \quad F_n(v_n) = (\Omega_n v_n \Omega_n) / v_n^2. \quad (\text{C.13})$$

which tell us that for untwisted representations the local maps F_n leave unchanged the spectrum of the generators of the Weyl algebras while this is not anymore true for the twisted ones.

In particular, these statements imply that for even chain and untwisted representations the SG transfer matrix is similar (up to the inhomogeneity flipping) to the τ_2 -transfer matrix while this is not anymore the case for twisted representations.

Remark 6. It is worth remarking that while the spectral problem of the SG model and the τ_2 -model are in general different the method here introduced to analyzed the SG spectrum can be extended to τ_2 -model. Such an extension is of sure interest as it should allow to get also for the τ_2 -model the strong statements of the simplicity and completeness of the transfer matrix spectrum.

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